

Nonholonomic Black Ring and Solitonic Solutions in Finsler and Extra Dimension Gravity Theories

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Abstract We study stationary configurations mimicking nonholonomic locally anisotropic black rings (for instance, with ellipsoidal polarizations and/or imbedded into solitonic backgrounds) in three/six dimensional pseudo-Finsler/Riemannian spacetimes. In the asymptotically flat limit, for holonomic configurations, a subclass of such spacetimes contains the set of five dimensional black ring solutions with regular rotating event horizon. For corresponding parameterizations, the metrics and connections define Finsler–Einstein geometries modeled on tangent bundles, or on nonholonomic (pseudo) Riemannian manifolds. In general, there are vacuum nonholonomic gravitational configurations which can not be generated in the limit of zero cosmological constant.

Keywords Pseudo-Finsler geometry · Nonholonomic manifolds and bundles · Nonlinear connections · Black rings

1 Introduction

There is a recent interest in (pseudo) Finsler geometry and applications to gravity [1–7], cosmology and astrophysics [8–11], see reviews of results and methods in [12, 13]. This paper is a partner of work [14], where Finsler black hole, ellipsoid and solitonic solutions

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were constructed for two classes of models of (pseudo) Finsler gravity on nonholonomic (pseudo) Riemannian manifolds and/or tangent bundles. We follow our anholonomic deformation method of constructing exact solutions in gravity and the aim of this letter is to study axisymmetric stationary solutions in higher dimensions and show how black ring configurations can be generated in Finsler gravity theories.¹

The most important property of the axisymmetric stationary solutions both in Finsler like theories and extra-dimension gravity is the fact that they admit event horizons with non-spherical topology which is in contrast to the four dimensional general relativity theory. In general, the topology of the event horizon can not be uniquely determined which provides a number of possible theoretical and experimental verifications of gravity theories and analogous models of classical and quantum interactions. In order to consider nonholonomic transforms of exact solutions in (pseudo) Finsler and/or extra dimension gravity with nontrivial topology and possible spacetime topology transitions, we shall use as ‘prime’ metrics certain classes of six dimensional metrics containing as imbedding five-dimensional black ring/hole configurations, for instance, having topology of \mathbb{S}^3 -sphere, or $\mathbb{S}^1 \times \mathbb{S}^2$ -torus.² The ‘target’ metrics generated by nonholonomic deforms will be constructed to possess the same, or different type topology, which for Finsler spaces positively can be more complicated and with a more ‘rich’ spacetime geometry because of existence of a nontrivial nonlinear connection (N-connection) structure.³

Assuming the existence of two additional commuting axial killing vector fields and the horizon topology of black ring $\mathbb{S}^1 \times \mathbb{S}^2$, it was found [17] that there is only one asymptotically flat black ring solution with a regular horizon which is the so-called Pomeransky–Sen’kov black ring [18]. There were also found many other black ring/object solutions (black Saturn/torus-ellipsoid configurations, di-ring/bi-ring etc.), see reviews and original results in [19–23] and Part II of monograph [7] where solutions with torus and ellipsoid nonholonomic configurations are investigated in details.

The discoveries that certain uniqueness black hole and non-hair theorems are violated in higher dimensions were regarded as very surprising and related to solutions with non-spherical horizon topology. We add to the list of such counterexamples of black hole fundamental theorems a new series of solutions generated by nonholonomic deformations and modeling Finsler configurations on higher dimension Einstein spaces and/or tangent bundles.

The paper is organized as follows: In Sect. 2, we fix a very general ansatz for a class of metrics with toroidal topology which are nonholonomically deformed on (pseudo) Finsler/Riemannian spacetimes. Such metrics are subjected to the condition to define Finsler–Einstein spacetimes as exact solutions of corresponding Einstein equations with cosmological constant. Section 3 is devoted to a class of 6-dimensional exact solutions with nontrivial cosmological constants. There are considered certain conditions when such solutions define nonholonomic deformations of black ring metrics and (pseudo) Finsler polarized black rings with possible ellipsoidal deformations and solitonic perturbations. There

¹The solutions analyzed in this work are different from the locally anisotropic (black) ellipsoid/torus configurations considered in Chaps. 10–12 of [7]. In this paper, our goal is to analyze black ring metrics and their nonholonomic deformations just for the (pseudo) Finsler spaces.

²In this paper, we shall use the term spacetime for a (pseudo) Riemannian/Finsler manifold enabled with a metric structure of signature \pm .

³We consider that our readers are familiar with the main geometric concepts of Lagrange-Finsler geometry modelled on tangent bundles [15, 16], or on (pseudo) Riemannian manifolds enabled with nonintegrable distributions (i.e. nonholonomic/N-anholonomic manifolds) [7, 13, 14].

are stated the conditions when Finsler type solutions transform into the Levi–Civita ones. In Sect. 4, we analyze (pseudo) Finsler stationary vacuum solutions for the so-called canonical distinguished connection in Finsler geometry and the Levi–Civita connection in extra dimension gravity. We show that for small nonholonomic deformations, it is possible to generate Finsler type black ring solutions with small polarizations of physical parameters. Finally, we provide some concluding remarks in Sect. 5.

2 Nonholonomic Ring Ansatz for Finsler–Einstein Spaces

2.1 Geometric Preliminaries

We can consider a six dimensional (6-d) manifold \mathbf{V} of necessary differentiability class. We endow it with a 3-dimensional non-integrable distribution \mathcal{D} . The pair $(\mathbf{V}, \mathcal{D})$ is called a nonholonomic manifold. We label the local coordinates $u = (x, y)$ on an open region $U \subset \mathbf{V}$ in the form $u^\alpha = (x^i, y^a)$ with indices $i, j, k, \dots = 1, 2, 3$ and $a, b, c, \dots = 4, 5, 6$.⁴ Let $e_{i'}$ be a local frame for \mathcal{D} such that $e_{i'} = e^{i'}_{i'} \frac{\partial}{\partial x^{i'}} + e^{a'}_{i'} \frac{\partial}{\partial y^{a'}}$ with $\det |e^{i'}_{i'}| \neq 0$. The Einstein convention on summation is applied. For any set of frame coefficients $e^{i'}_{i'}$, we can define $e_i^{j'} \equiv (e^{-1})^i_{j'}$, following formulas $e^{i'}_{i'} e_i^{j'} = \delta_i^{j'}$, where $\delta_i^{j'}$ is the Kronecker symbol (such frame and coframe coefficients can be defined, in general, on any U).

Expressing $\frac{\partial}{\partial x^i} = e_i^{j'} e_{j'} + e_i^{a'} \frac{\partial}{\partial y^{a'}}$, taking $\mathbf{e}_i = e_i^{j'} e_{j'}$ as a new local frame on \mathcal{D} considering that $e_a^{a'} = \delta_a^{a'}$ and $e_i^a \equiv N_i^a(x, y)$, we get $\mathbf{e}_i = \delta_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a}$.⁵ We may complete \mathbf{e}_i to a local frame $\mathbf{e}_\alpha = (\mathbf{e}_i, e_a)$, for $e_a = \partial_a \equiv \frac{\partial}{\partial y^a}$, on \mathbf{V} . This fact suggests to consider also on \mathbf{V} the distribution $\tilde{\mathcal{D}}$ locally spanned by ∂_a . It is supplementary to \mathcal{D} and locally integrable. If a change of coordinates $(x^i, y^a) \rightarrow (x^{i'}(x^i, y^a), y^{a'}(x^i, y^a))$ is performed, the formula $\frac{\partial}{\partial y^a} = \frac{\partial x^{i'}}{\partial y^a} \frac{\partial}{\partial x^{i'}} + \frac{\partial y^{a'}}{\partial y^a} \frac{\partial}{\partial y^{a'}}$ has to simplify to $\frac{\partial}{\partial y^a} = \frac{\partial y^{a'}}{\partial y^a} \frac{\partial}{\partial y^{a'}}$ and thus $\frac{\partial x^{i'}}{\partial y^a} = 0$. This equality and the general condition $\delta_j = e_j^{j'} \delta_{j'}$ with $\det |e_j^{j'}| \neq 0$, where $\delta_{j'} = \frac{\partial}{\partial x^{j'}} - N_{j'}^a(u) \frac{\partial}{\partial y^{a'}}$, imply $e_j^{j'} = \frac{\partial x^{j'}}{\partial x^j}$ and $\frac{\partial x^{j'}}{\partial x^j} N_{j'}^a = \frac{\partial y^{a'}}{\partial y^a} N_j^a - \frac{\partial y^{a'}}{\partial x^j}$. Alternatively, we may say that \mathbf{V} is a foliated manifold with foliation determined by the distribution $\tilde{\mathcal{D}}$, called also structural distribution, and having \mathcal{D} as a transversal distribution.

In this work the nonholonomic aspects will be more important and the stress will be upon the distribution \mathcal{D} . In particular, \mathbf{V} can be the total spaces, for instance, of a submersion over a 3-d manifold M , or (more particular) a vector bundle E over M , a principal bundle over M , or the (co) tangent bundle TM (T^*M). The general geometrical framework just described, terms and techniques from Finsler and Lagrange geometries [7, 12, 13, 15, 16, 31] will be used. Thus we call a decomposition $TV = \mathcal{D} \oplus \tilde{\mathcal{D}}$ as a nonlinear connection (N-connection) structure \mathbf{N} on \mathbf{V} with local coefficients $N_i^a(u)$.⁶ For $\mathbf{V} = TM$, we can write $TTM = hTM \oplus vTM$ with certain natural “horizontal” (h)

⁴In a similar form we shall use ‘primed’ indices and coordinates, $u^{a'} = (x^{i'}, y^{a'})$, when $i', j', k', \dots = 1, 2, 3$ and $a', b', c', \dots = 4, 5, 6$.

⁵In our works, boldface symbols are used for nonholonomic manifolds/bundles (nonholonomic spaces) and geometric objects on such spaces; we can consider that δ_i are the so-called N-elongated partial derivatives in Finsler geometry and generalizations.

⁶In this work, we shall use the so-called canonical Cartan N-connection and the distinguished connection, d-connection, see details in [13, 16].

and “vertical” (v) decompositions. In such a case, a N-connection structure can be derived canonically from any regular Lagrangian $L : TM \rightarrow \mathbb{R}$ (or for some more particular homogeneous on variables y^a cases, from a fundamental (generating) Finsler function $F(x, y)$, when $F(x, \beta y) = \beta F(x, y)$, $\beta > 0$). This allows us to construct various models of Lagrange–Finsler geometry and gravity theories on TM , when the variables y^a are of “velocity/momentum” type. Alternatively, we can work on a (pseudo) Riemannian nonholonomic manifold \mathbf{V} endowed with a corresponding metric structure $\mathbf{g} = \{g_{\alpha\beta}(u)\}$. For this class of nonholonomic geometry and gravity theories, any types of conventional distributions \mathcal{D} and $\tilde{\mathcal{D}}$ can be introduced on \mathbf{V} on convenience, for instance, with the aim to define a general geometric method of constructing exact solutions in general relativity and various modifications of Ricci flow theory [7, 12, 13, 26–28], when y^a can be treated as certain general nonholonomically constrained variables/coordinates (they can be defined even on usual Einstein spaces of arbitrary dimension).

2.2 Ring Type Einstein and Finsler–Einstein Spaces

We shall use a ‘prime’ metric on \mathbf{V}

$$\check{\mathbf{g}} = \check{g}_i(x^k)dx^i \otimes dx^i + \check{h}_a(x^k)\check{\mathbf{e}}^a \otimes \check{\mathbf{e}}^a, \tag{1}$$

$$\check{\mathbf{e}}^a = dy^a + \check{N}_i^a(x^k)dx^i.$$

For certain parameterizations,⁷ this metric contains as a trivial embedding (with extension on coordinate y^6 , for a term of type $\pm(dy^6)^2$ in the metric quadratic form, into a 6-d pseudo-Riemannian space) the 5-d black ring metric with a dipole, or with rotation in the \mathbb{S}^2 [24]. The 5-d part of (1), in coordinates (x^i, y^4, y^5) , is prescribed to be conformally proportional, with the factor $(x - y)^2 F(y)/\check{R}^2 F(x)G(y)$, to the black ring metric

$$\begin{aligned} {}^{br}\mathbf{g} = & \check{R}^2 \frac{F(x)}{(x - y)^2} \left[-\frac{G(y)}{F(y)} d\psi \otimes d\psi + G^{-1}(x)dx \otimes dx - G^{-1}(y)dy \otimes dy \right] \\ & + \check{R}^2 \frac{G(x)}{(x - y)^2} d\phi \otimes d\phi \\ & - \frac{F(y)}{F(x)} \left(dt - C\check{R}\frac{1+y}{F(y)}d\psi \right) \otimes \left(dt - C\check{R}\frac{1+y}{F(y)}d\psi \right). \end{aligned} \tag{2}$$

The coefficients of this metric are stated by a class of functions and constants: for instance, $F(x) = 1 + \check{\lambda}x$ and $G(y) = (1 - y^2)(1 + \nu y)$ for $\check{R} = const$ and

$$C = \sqrt{\check{\lambda}(\check{\lambda} - \nu)(1 + \check{\lambda})/(1 - \check{\lambda})}$$

is determined by certain dimensionless parameters $\check{\lambda}$ and ν satisfying the condition $0 < \nu \leq \check{\lambda} < 1$ (here we note that our notations are different from those used in [19]). The metric (2) defines an exact solution of Einstein equations for the Levi–Civita in 5-d gravity.

⁷ $\check{g}_1 = -1, \check{g}_2 = \check{g}_2(x^2, x^3), \check{g}_3 = \check{g}_3(x^2, x^3), \check{h}_6(x^k) = \pm 1$ and some vanishing N-connection coefficients with respect to a corresponding local coordinate basis, $\check{N}_i^4 = 0, \check{N}_i^6 = 0, \check{N}_{2,3}^5 = 0$, but $\check{N}_i^5 \neq 0$, and for local coordinates $x^i = x^i(\psi, x, y), y^4 = \phi, y^5 = t, y^6 = y^6$.

Using nonholonomic deformations of the locally isotropic black ring metric (1) with $g_i(x^k) = \eta_i(x^k)\check{g}_i$ and $h_{\hat{a}} = \eta_{\hat{a}}(x^k, y^4)\check{h}_{\hat{a}}$ and some nontrivial $N_i^4 = w_i(x^k, y^4)$ and $N_i^5 = n_i(x^k, y^4)$, but $N_i^6 = 0$, when indices with “hats” run values $\hat{i}, \hat{k}, \dots = 2, 3$ and $\hat{a}, \hat{b}, \dots = 4, 5$, we generate a metric of type

$$\mathbf{g} = g_i(x^{\hat{k}})e^i \otimes e^j + h_a(x^k, y^4)\mathbf{e}^a \otimes \mathbf{e}^b, \tag{3}$$

$$\mathbf{e}^a = dy^a + N_i^a(x^k, y^4)dx^i, \tag{4}$$

when values $\mathbf{g}_{\alpha'\beta'} = [g_i, h_a]$ with $g_i = -1$ and $h_6 = \pm 1$ are related by frame transforms

$$\mathbf{g}_{\alpha'\beta'}e^{\alpha'}(x^k, y^a)e^{\beta'}(x^k, y^a) = \mathbf{f}_{\alpha\beta}(x^k, y^a) \tag{5}$$

to a (pseudo-Finsler) metric $\mathbf{f}_{\alpha\beta} = [f_{ij}, f_{ab}]$ and corresponding N-adapted dual canonical basis ${}^c\mathbf{e}^\alpha = (dx^i, {}^c\mathbf{e}^a = dy^a + {}^cN_i^a(x^k, y^a)dx^i)$ defined by the so-called canonical Cartan N-connection in Finsler geometry, see details in [12–16].⁸ For any given values $\mathbf{g}_{\alpha'\beta'}$ and $\mathbf{f}_{\alpha\beta}$, we have to solve a system of quadratic algebraic equations (5) in order to determine the unknown variables $e^{\alpha'}$. How to define in explicit form such frame coefficients (vierbeins) and coordinates in 4-d we discuss in [12, 14], but the algebraic computations are similar for 6-d. As a matter of principle, any (pseudo) Riemannian metric $\mathbf{g}_{\alpha'\beta'}$ can be expressed in Finsler variables as a metric $\mathbf{f}_{\alpha\beta}$, up to a trivial imbedding into an even dimension, (the inverse statement also holds true), if a N-connection structure is prescribed on \mathbf{V} . The standard (pseudo) Finsler geometric and gravitational models (for instance, those from [15, 16]) are constructed on $\mathbf{V} = TM$, when ‘fiber’ variables y^a are treated as velocities, but Finsler like structures/variables can be defined also in the Einstein gravity and string generalizations [7, 12, 13] when y^a are considered as certain nonholonomically constrained coordinates on (pseudo) Riemannian manifolds with possible torsion generalizations.

In this work, we shall analyze a class of 6-d metrics (or 3-d Finsler metrics) defining Finsler–Einstein spaces as exact solutions of the Einstein equations,

$$\hat{R}^i_j = \lambda\delta^i_j, \quad \hat{R}^a_b = \lambda\delta^a_b, \quad \hat{R}_{i\hat{a}} = \hat{R}_{\hat{a}i} = 0 \tag{6}$$

⁸Any (pseudo) Finsler metric $\mathbf{f} = \{\mathbf{f}_{\alpha\beta}\}$ can be parametrized in the canonical Sasaki form

$$\mathbf{f} = f_{ij}dx^i \otimes dx^j + f_{ab}{}^c\mathbf{e}^a \otimes {}^c\mathbf{e}^b, \quad {}^c\mathbf{e}^a = dy^a + {}^cN_i^a dx^i,$$

where the (pseudo) Finsler configuration is defined by 1) a fundamental real Finsler (generating) function $F(u) = F(x, y) = F(x^i, y^a) > 0$ if $y \neq 0$ and homogeneous of type $F(x, \lambda y) = |\lambda|F(x, y)$, for any nonzero $\lambda \in \mathbb{R}$, with positively definite Hessian $f_{ab} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^a \partial y^b}$, when $\det|f_{ab}| \neq 0$. The Cartan canonical N-connection structure ${}^c\mathbf{N} = \{{}^cN_i^a\}$ is completely defined by an effective Lagrangian $L = F^2$ in such a form that the corresponding semi-spray configuration is defined by nonlinear geodesic equations being equivalent to the Euler–Lagrange equations for L (see details, for instance, in [7, 13, 16]; for “pseudo” configurations, this mechanical analogy is a formal one, with some “imaginary” coordinates [14]). One defines ${}^cN_i^a = \frac{\partial G^a}{\partial y^{3+i}}$ for $G^a = \frac{1}{4} f^a{}_{3+i} (\frac{\partial^2 L}{\partial y^{3+i} \partial x^k}) y^{3+k} - \frac{\partial L}{\partial x^i}$, where f^{ab} is inverse to f_{ab} and respective contractions of horizontal (h) and vertical (v) indices, i, j, \dots and a, b, \dots , are performed following the rule: we can write, for instance, an up v -index a as $a = 3 + i$ and contract it with a low index $i = 1, 2, 3$. In brief, for spaces of even dimension, we shall write y^i instead of y^{3+i} , or y^a .

where $\hat{\mathbf{R}}_{\alpha\beta} = \{\hat{R}_{ij}, \hat{R}_{ia}, \hat{R}_{ai}, \hat{R}_{ab}\}$ are the components of the Ricci tensor computed for the canonical distinguished connection (d-connection) $\hat{\mathbf{D}}$, see details in [7, 12–14],⁹ λ is a cosmological constant, and δ^i_j , for instance, is the Kronecker symbol. Solutions of non-holonomic equations (6) are typical ones for the Finsler gravity with metric compatible d-connections (in our case, $\hat{\mathbf{D}}\mathbf{f} = \mathbf{0}$) modelled in various types of (super) string, noncommutative and other generalizations, see reviews of results in references in [6, 7, 13].

Imposing additional nonholonomic constraints on coefficients of metrics solving the system (6) but when the distortion of the Ricci tensor (under nonholonomic deforms from one linear connection to another) is zero, we select a more restricted class of Finsler–Einstein configurations defining exact solutions in the 6-d generalization of the Einstein gravity, with the Ricci tensor ${}_iR_{\alpha\beta}$ for the Levi–Civita connection ∇ , i.e. solutions of

$${}_iR_{\alpha\beta} = \lambda g_{\alpha\beta}. \tag{7}$$

There are Finsler–Einstein configurations with $\hat{\mathbf{R}}_{\alpha\beta} = {}_iR_{\alpha\beta}$ but, in general, such spaces have different curvature tensors because they are defined by different linear connections.

The goal of this work is to construct and analyze possible physical implications of a class of exact solutions of (6) and (7) with $\lambda \neq 0$, or $\lambda = 0$, with metrics of type (3) containing in certain limits the black ring metric (2). The new classes of nonholonomically deformed black rings will be considered for the (pseudo) Finsler gravity on tangent bundle and/or for usual 6-d Einstein spacetimes.

3 6-d Solutions with Nontrivial Cosmological Constant

In this section we consider two classes of exact solutions of (non) holonomic Einstein equations with $\lambda \neq 0$ generated by corresponding conformal transform of the 5-d black ring metric (2), nonholonomic deformations and imbedding into a 6-d (pseudo) Riemannian, or in a 3-d (pseudo) Finsler spacetime.

⁹The coefficients of $\hat{\mathbf{D}}$ with respect to a “N-elongated” frames of type (4) are computed $\hat{\Gamma}^\gamma_{\alpha\beta} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^i_{jc}, \hat{C}^a_{bc})$ with

$$\begin{aligned} \hat{L}^i_{jk} &= \frac{1}{2} g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \\ \hat{L}^a_{bk} &= e_b (N_k^a) + \frac{1}{2} h^{ac} (e_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d), \\ \hat{C}^i_{jc} &= \frac{1}{2} g^{ik} e_c g_{jk}, \quad \hat{C}^a_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_c h_{cd} - e_d h_{bc}). \end{aligned}$$

Any geometric construction for the canonical d-connection $\hat{\mathbf{D}}$ can be re-defined equivalently into a similar one with the Levi–Civita connection $\nabla = \{\Gamma^\alpha_{\beta\gamma}\}$, and inversely following the formula ${}_i\Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} + {}_iZ^\gamma_{\alpha\beta}$, where the distortion tensor ${}_iZ^\gamma_{\alpha\beta}$ is computed as

$$\begin{aligned} {}_iZ^a_{jk} &= -C^i_{jb} g_{ik} h^{ab} - \frac{1}{2} \Omega^a_{jk}, \quad {}_iZ^i_{bk} = \frac{1}{2} \Omega^c_{jk} h_{cb} g^{ji} - \Xi^{ih}_{jk} C^j_{hb}, \\ {}_iZ^a_{bk} &= +\Xi^{ab\circ} L^c_{bk}, \quad {}_iZ^i_{kb} = \frac{1}{2} \Omega^a_{jk} h_{cb} g^{ji} + \Xi^{ih}_{jk} C^j_{hb}, \quad {}_iZ^i_{jk} = 0, \\ {}_iZ^a_{jb} &= -\Xi^{ad\circ} L^c_{dj}, \quad {}_iZ^a_{bc} = 0, \quad {}_iZ^i_{ab} = -\frac{g^{ij}}{2} \left[\circ L^c_{aj} h_{cb} + \circ L^c_{bj} h_{ca} \right], \end{aligned}$$

for $\Xi^{ih}_{jk} = \frac{1}{2} (\delta^i_j \delta_k^h - g_{jk} g^{ih})$, $\pm \Xi^{ab}_{cd} = \frac{1}{2} (\delta^a_c \delta_d^b + h_{cd} h^{ab})$ and $\circ L^c_{aj} = L^c_{aj} - e_a (N^c_j)$.

3.1 Ansatz for Prime and Target Metrics

Let us consider a prime metric

$$\begin{aligned}
 {}^{br}_{cf} \mathbf{g} = & -d\psi \otimes d\psi + \frac{(x-y)^2 F(y)}{\check{R}^2 F(x) G(y)} \left[\frac{dx \otimes dx}{G(x)} - \frac{dy \otimes dy}{G(y)} \right] \\
 & + \frac{{}^1 G(x) F(y)}{F(x) G(y)} d\phi \otimes d\phi \\
 & - \frac{(x-y)^2 F^2(y)}{\check{R}^2 F^2(x) G(y)} \left(dt - C \check{R} \frac{1+y}{F(y)} d\psi \right) \otimes \left(dt - C \check{R} \frac{1+y}{F(y)} d\psi \right), \tag{8}
 \end{aligned}$$

which multiplied to the conformal factor $\check{R}^2 \frac{F(x)}{(x-y)^2} \frac{G(y)}{F(y)}$ and for ${}^1 G(x) = G(x)$ is just the black ring metric (2). We introduce variables

$$\begin{aligned}
 x^1 = \psi, \quad x^2 = \int dx / \sqrt{|G(x)|}, \quad x^3 = \int dy / \sqrt{|G(y)|}, \\
 y^4 = \phi, \quad y^5 = t, \quad y^6 = y^6
 \end{aligned}$$

and label the metric coefficients as

$$\begin{aligned}
 \check{g}_2(x^2, x^3) &= \frac{[x(x^2) - y(x^3)]^2 F(x^3)}{\check{R}^2 F(x^2) G(x^3)} = -\check{g}_3(x^2, x^3), \quad \check{g}_1 = -1, \quad \check{h}_6 = \pm 1, \\
 \check{h}_4(x^2, x^3) &= \frac{{}^1 G(x^2) F(x^3)}{F(x^2) G(x^3)}, \quad \check{h}_5(x^2, x^3) = -\frac{[x(x^2) - y(x^3)]^2 F(x^3)}{\check{R}^2 F(x^2) G(x^3)}, \\
 \check{N}_1^5(x^3) &= C \check{R} \frac{1+y(x^3)}{F(x^3)}, \quad \check{N}_2^5 = \check{N}_3^5 = 0, \quad \check{N}_i^4 = 0, \quad \check{N}_i^6 = 0.
 \end{aligned}$$

Imbedding ${}^{br}_{cf} \mathbf{g}$ into a 6-d spacetime, we get a metric

$$\begin{aligned}
 \check{\mathbf{g}} = & \check{g}_1 dx^1 \otimes dx^1 + \check{g}_2 dx^2 \otimes dx^2 + \check{g}_3 dx^3 \otimes dx^3 \\
 & + \check{h}_4 \check{\mathbf{e}}^4 \otimes \check{\mathbf{e}}^4 + \check{h}_5 \check{\mathbf{e}}^5 \otimes \check{\mathbf{e}}^5 + \check{h}_6 \check{\mathbf{e}}^6 \otimes \check{\mathbf{e}}^6, \tag{9} \\
 \check{\mathbf{e}}^4 = & dy^4, \quad \check{\mathbf{e}}^5 = dy^5 + \check{N}_1^5(x^3) dx^1, \quad \check{\mathbf{e}}^6 = dy^6,
 \end{aligned}$$

which, in general, is not a solution of field equations for a gravitational model. We search for some classes of metrics generated by nonholonomic deformations with “polarization” multiples $\eta_\alpha = [\eta_i(u^\beta), \eta_a(u^\beta)]$, when $g_\alpha = \eta_\alpha \check{g}_\alpha = [g_i = \eta_i \check{g}_i, h_a = \eta_a \check{h}_a]$, and modified N-connection coefficients $N_i^a(u^\beta)$ resulting in exact solutions of (6) and/or (7).

For

$$\begin{aligned}
 \eta_1 = 1, \quad \eta_2 \check{g}_2 = \epsilon_2 e^{\phi(x^2, x^3)}, \quad \eta_3 \check{g}_3 = \epsilon_3 e^{\phi(x^2, x^3)}, \\
 \eta_6 = 1, \quad h_4(x^k, y^4) = \eta_4 \check{h}_4, \quad h_5(x^k, y^4) = \eta_5 \check{h}_5,
 \end{aligned}$$

where $\epsilon_\alpha = \pm 1$ for chosen signatures, and

$$N_i^4 = w_i(x^k, y^4), \quad N_i^5 = n_i(x^k, y^4), \quad N_i^6 = 0,$$

we get a class of ‘target’ generic off-diagonal metrics¹⁰

$$\begin{aligned} \lambda \mathbf{g} &= -dx^1 \otimes dx^1 + e^{\varphi(x^2, x^3)} [\epsilon_2 dx^2 \otimes dx^2 + \epsilon_3 dx^3 \otimes dx^3] \\ &\quad + h_4(x^k, y^4) \mathbf{e}^4 \otimes \mathbf{e}^4 + h_5(x^k, y^4) \mathbf{e}^5 \otimes \mathbf{e}^5 \pm dy^6 \otimes dy^6, \\ \mathbf{e}^4 &= dy^4 + w_i(x^k, y^4) dx^i, \quad \mathbf{e}^5 = dy^5 + n_i(x^k, y^4) dx^i, \end{aligned} \tag{10}$$

with the coefficients will be defined in next sections such a way that these metrics are exact solutions of the Einstein equations in (pseudo) Riemannian/Finsler gravity.

3.2 (Pseudo) Finsler Polarized Black Rings

We shall use brief denotations for some partial derivatives like $a^\bullet = \partial a / \partial x^2$, $a' = \partial a / \partial x^3$, $a^* = \partial a / \partial y^4$.

3.2.1 Stationary Nonholonomic Deformations of Black Ring Metrics

By straightforward computations,¹¹ we can verify that a metric of type (10) generates an exact solution of the Einstein equations (6) if the coefficients are determined by any functions satisfying (respectively) the conditions:

$$\begin{aligned} \epsilon_2 \varphi^{\bullet\bullet}(x^{\hat{k}}) + \epsilon_3 \varphi''(x^{\hat{k}}) &= -2\epsilon_2 \epsilon_3 \lambda; \\ h_4 &= \pm \frac{(\varphi^*)^2}{4\lambda} e^{-2\varphi(x^i)}, \quad h_5 = \mp \frac{1}{4\lambda} e^{2(\varphi - \varphi(x^i))}; \\ w_i &= -\partial_i \varphi / \varphi^*; \\ n_i &= {}^1n_i(x^k) + {}^2n_i(x^k) \int dy^4 (\varphi^*)^2 e^{-2(\varphi - \varphi(x^i))} \\ &= \begin{cases} {}^1n_i(x^k) + {}^2n_i(x^k) \int dy^4 e^{-4\varphi} \frac{(h_5^*)^2}{h_5}, & \text{if } n_i^* \neq 0; \\ {}^1n_i(x^k), & \text{if } n_i^* = 0; \end{cases} \end{aligned} \tag{11}$$

for any nonzero $h_{\hat{a}}$ and $h_{\hat{a}}^*$ and (integration) functions ${}^1n_i(x^k)$, ${}^2n_i(x^k)$, a generating function $\varphi(x^i, y^4)$, and ${}^0\varphi(x^i)$ to be determined from certain boundary conditions for a fixed system of coordinates with $x^{\hat{k}} = \{x^2, x^3\}$.

There are two classes of solutions (11) constructed for a nontrivial λ . The first one is singular for $\lambda \rightarrow 0$ if we choose a generation function $\varphi(x^i, y^4)$ not depending on λ . It is possible to eliminate such singularities for certain parametric dependencies of type $\varphi(\lambda, x^i, y^4)$, when the resulting metric and N-connection coefficients are not singular on λ .

3.2.2 Small Eccentricity Ellipsoid Polarizations and Ring Deformations

It is not clear what kind of physical implications one may have exact solutions with general coefficients of type (11). But it is possible to extract a subclass of solutions decomposed

¹⁰Which can not be diagonalized, in general, by any coordinate transform.

¹¹For simplicity, we omit such computations in this work which are similar to those presented in Sect. 2.7, pp. 143–150, in [7], see also reviews on the anholonomic deformation method of constructing exact solutions in [6, 12, 13].

on a small parameter ε which will define certain small nonholonomic deformations of the conformally transformed black ring metric (8) on a 6-d (pseudo) Riemannian space.¹²

We chose a polarization function (i.e. a nonlinear self-polarization of gravitational vacuum with cosmological constant λ) of type

$$\eta_4 = \eta(x^i, \phi, \varepsilon), \tag{12}$$

where $\eta(x^i, \phi, \varepsilon)$ is any linear on ε function depending in general on variables x^i and ϕ . Introducing $h_4 = \eta_4 \check{h}_4$ and $h_5 = \eta_5 \check{h}_5$ into the second line of equations (11), integrating on ϕ and stating the integration function ${}^0\varphi(x^i) = 1$, we get

$$\begin{aligned} \varphi &= 2\sqrt{|\lambda \check{h}_4|} \int d\phi \sqrt{|\eta(x^i, \phi, \varepsilon)|}, \\ \eta_5 &= \mp \frac{1}{4\lambda \check{h}_5} \exp\left[4\sqrt{|\lambda \check{h}_4|} \int d\phi \sqrt{|\eta(x^i, \phi, \varepsilon)|}\right]. \end{aligned} \tag{13}$$

Haven defined polarizations η_4 and η_5 , we can compute the coefficients h_4 and h_5 , which can be used for computing the nontrivial N-connection coefficients

$$w_i = -\partial_i \varphi / \varphi^* \quad \text{and} \quad n_i = {}^1n_i(x^k) + {}^2n_i(x^k) \int (\varphi^*)^2 e^{-2\varphi} d\phi.$$

We choose ${}^1n_i(x^k) = C \check{R} \frac{1+y(x^3)}{F(x^3)}$ in order to have certain similarity with the prime metric (8).

Further approximations are possible, for instance, for $\omega(x^2, x^3) = \omega_0 x^2$ and $\eta(x^i, \phi, \varepsilon)$ when

$$\eta_4 = 1 + \varepsilon \check{\eta}_4(x^i, \phi, \varepsilon), \quad \eta_5 = [1 + \varepsilon \cos(\omega_0 x^2)] [1 + \varepsilon \check{\eta}_5(x^i, \phi, \varepsilon)]$$

and $\eta_2 \check{g}_2 = \varepsilon_2 e^{\phi(x^2, x^3)}$, $\eta_3 \check{g}_3 = \varepsilon_3 e^{\phi(x^2, x^3)}$ are determined by any solution ϕ of the first equation in (11) and $\check{\eta}_5$ is computed using (13). This generates a subclass of Finsler–Einstein spaces parametrized by metrics of type

$$\begin{aligned} {}^\lambda_\varepsilon \mathbf{g} &= -dx^1 \otimes dx^1 + e^{\phi(x^2, x^3)} [\varepsilon_2 dx^2 \otimes dx^2 + \varepsilon_3 dx^3 \otimes dx^3] \\ &+ [1 + \varepsilon \check{\eta}_4] \check{h}_4 \mathbf{e}^4 \otimes \mathbf{e}^4 \\ &+ [1 + \varepsilon \cos(\omega_0 x^2)] [1 + \varepsilon \check{\eta}_5] \check{h}_5 \mathbf{e}^5 \otimes \mathbf{e}^5 \pm dy^6 \otimes dy^6, \\ \mathbf{e}^4 &= dy^4 + w_i(x^k, y^4) dx^i, \quad \mathbf{e}^5 = dy^5 + n_i(x^k, y^4) dx^i. \end{aligned} \tag{14}$$

The metric (14) with coefficients computed with respect to the dual frame of reference $\mathbf{e}^\alpha = (\eta_i dx^i, \mathbf{e}^4, \sqrt{|\eta_5|} \mathbf{e}^5, dy^6)$ is similar to a nonholonomically polarized 5-d black ring imbedded self-consistently in a 6-dimensional spacetime. Really, the multiple $[1 + \varepsilon \cos(\omega_0 x^2)]$ before \check{h}_5 determines an elliptic polarization with eccentricity ε when

$$\check{r} \sim \check{R} / [1 + \varepsilon \cos(\omega_0 x^2)] \tag{15}$$

¹²Such a solution will be an exact solution because this type of series decompositions are not on coordinate variables but for certain fixed small parameters which can be always defined for metrics with Killing symmetries, see details in [6, 12, 13].

describes an ellipse with coordinate/anisotropy ($\omega_0 x^2$). The multiple $[1 + \varepsilon \tilde{\eta}_4]$ before \check{h}_4 can be included as a polarization of function 1G in (8), when ${}^1G(x^2) \rightarrow G(x^2)\eta_4(x^i, \phi, \varepsilon) \sim G[1 + \varepsilon \tilde{\eta}_4]$. Such small on ε nonholonomic deformations of the thin black rings seem to be stable but may result in unstable solutions for minimally spinning and/or fat black rings, see details for holonomic ring configurations in [25].¹³

One should emphasize that for small ε there are preserved the same type of singularities of solutions and horizons as for the usual black ring metrics but with certain polarizations of constants and additional smooth terms, for instance, to the curvature tensor and other tensors/connections, all being proportional to ε . More general nonholonomic deformations result not only in possible instabilities but, in general, in various types of “non-ring” stationary configurations.

3.2.3 Solitonic Perturbations of Nonholonomic Black Rings

It is possible to consider self-consistent imbedding of the metric (14) into nontrivial backgrounds similarly as it was done for solitonic “motions” of black holes in five dimensional gravity [27] and/or in nonholonomic Ricci flows of exact solutions in gravity [28]. There are very different classes of solutions for nonholonomic black ring—solitonic configurations: (1) black rings are on “huge” stationary solitonic configurations, when the resulting general solution is not of ring type and (2) certain “small” stationary solitonic distributions preserve the black ring character for such generic off-diagonal metrics.

In the first case, for (12), we can use a static three dimensional solitonic distribution $\eta(x^2, x^3, \phi)$ defined as a solution of solitonic equation¹⁴

$$\eta^{**} + \varepsilon(\eta' + 6\eta\eta' + \eta^{***})^* = 0, \quad \varepsilon = \pm 1. \tag{16}$$

This induces solitonic polarizations for (13) and N-connection coefficients w_i and n_i obtained by integrating on ϕ some types of functionals depending of $\varphi(x^2, x^3, \phi)$ and defining solitonic distributions. The resulting metric is of type (14) is constructed as a stationary superposition of 3-d solitonic distributions for 5-d and 6-d Finsler–Einstein spacetimes when some multiples in the metric coefficients originate from former black ring coefficients of metrics. We need to perform a more special analysis on stability and physical meaning of such solutions for more special cases of distributions (this is out of scope of our paper). Here we emphasize that it is obvious that, in general, certain solitonic hierarchies and related conservation laws can be always associated (both for nonholonomic pseudo-Riemannian and pseudo-Finsler configurations), see details in [29, 30].

In the second case, considering that $\eta \sim 1 + \varepsilon \tilde{\eta}_4(x^2, x^3, \phi, \varepsilon)$, where $\tilde{\eta}_4$ is a solution of (16), we preserve the black ring character of the class of solutions even some physical parameters became solitonically polarized as we discuss below in Sect. 4.2. Such solutions in (pseudo) Finsler gravity have positively physical interpretations as black rings with locally anisotropic polarizations of the metric coefficients, relevant constants and parameters. But in this case, not only such polarizations distinguish these classes of solutions. The anisotropic coordinate ϕ is of “fiber” type in a tangent bundle, i.e. a special type of velocity-coordinate. The existence of such Finsler type black rings would be a topological evidence for broken

¹³The conditions of stability for nonholonomic ring configurations should be analyzed separately for a fixed class of nonholonomic distributions like in the case of black ellipsoids provided, for instance, in [6, 26].

¹⁴We can introduce instead such an equation any type of three/two dimensional solitonic or other nonlinear wave equations.

local Lorentz invariance. This class of solutions is not for the Levi–Civita connection, but for the canonical d-connection. Such Finsler–Einstein spaces with nonholonomically deformed Lorentz invariance (and broken local special relativity theory, double special relativity models etc. [2, 3, 11]) can be described in terms of canonical quadratic metric forms and N- and d-connection structures defined for a model of Finsler geometry on tangent bundle, as we show in footnote 8 and Sect. 3.4.

Finally, in this section, we emphasize that if we model a pseudo-Finsler geometry on a pseudo-Riemannian 6-d spacetime with the “fiber” coordinates treated as extra-dimensional extensions of a 3-d gravity model, we get an Einstein–Cartan 6-d spacetime with an effective torsion induced by some off-diagonal metric terms and as a corresponding nonholonomic frame effect. All geometric and physical constructions can be redefined in terms of the Levi–Civita connection, but nevertheless, the solutions will be those for the canonical d-connection. The above class of solutions, for such models, will describe certain possible nontrivial topological evidences, for instance, nonholonomic black rings, induced from extra-dimension gravity.

3.3 Nonholonomic Deformations with the Levi–Civita Connection

It is possible to constrain the integral varieties of solutions of the Einstein equations for the canonical d-connection (6), parametrized by an ansatz (10), or (14), in such a way that for certain values of coefficients the corresponding metric will be also a solution of (7) for the Levi–Civita connection. This is possible with respect to certain classes of N-adapted bases when the distortion of the Ricci tensor vanishes for certain nonholonomic configurations (see details on such constructions in [12–14]).

By straightforward computations, we can verify that we can extract from the above mentioned ansatz an exact 4-d solution (contained as a trivial imbedding in 6-d) in Einstein gravity with cosmological constant λ if the coefficients of the metric and N-connection are subjected to the conditions:

$$\begin{aligned}
 (2e^{2\varphi}\varphi - \lambda)(\varphi^*)^2 &= 0, & \varphi &\neq 0, & \varphi^* &\neq 0; \\
 w_2 w_3 \left(\ln \left| \frac{w_2}{w_3} \right| \right)^* &= w_3^\bullet - w_2', & w_i^* &\neq 0, & w_1 &= 0; \\
 w_3^\bullet - w_2' &= 0, & \text{if } w_i^* &= 0 \text{ and } w_1 = 0; \\
 {}^1 n_2'(x^k) - {}^1 n_3^\bullet(x^k) &= 0, & \text{if } n_i^* &= 0 \text{ and } n_1 = 0,
 \end{aligned}
 \tag{17}$$

which holds for any $\varphi(x^i, y^4) = \ln |h_3^*/\sqrt{|h_4 h_5|}| = const$ if we include configurations with $\varphi^* = 0$.

We can consider metrics of type (10) with coefficients of class (17) formally extended to 5-d and 6-d with certain nontrivial values of g_1 and/or h_5 which will contain as an imbedding the black ring metric (2) and its conformal transforms on variables x and y and various types of nonholonomic deformations. Such solutions describe certain nonholonomic black ring configurations for small on ε deformations like in (14). Nevertheless, they are different from those considered in [17–25]. In our case, the nonholonomic distributions are nontrivial, and induced by a nonzero cosmological constant, which polarize physical parameters and may state, for instance, a stationary solitonic background.

3.4 (Pseudo) Finsler Configurations

For a general (pseudo) Finsler configuration, the class of metrics (10) with coefficients (11) expressed in Finsler variables as a metric $f_{\alpha\beta}$, see formulas (5), depends formally on all six variables x^i and y^a and the imbedding into a 6-d (pseudo) Riemannian spacetime is not trivial. Nevertheless, this class of Finsler–Einstein spaces contains two Killing vectors because in certain systems of coordinates such metrics do not depend on variables y^5 and y^6 in explicit form. Such metrics are stationary because the coefficients do not depend on variable $y^5 = t$.

For any $f_{\alpha\beta} = e^{\alpha'}_{\alpha} e^{\beta'}_{\beta} g_{\alpha'\beta'}$ corresponding to a canonical Finsler type parametrization of metric and connections, see footnote 8, we can write in explicit h- and v-components

$$f_{ij} = e^{i'}_i e^{j'}_j g_{i'j'} \quad \text{and} \quad f_{ab} = e^{a'}_a e^{b'}_b g_{a'b'}, \tag{18}$$

$$N_{i'}^{a'} = e^i_{i'} e^{a'}_a N_i^a, \quad \text{or} \quad {}^c N_i^a = e^{i'}_i e^a_{a'} N_{i'}^{a'}, \tag{19}$$

were, for instance, $e^a_{a'}$ is inverse to $e^{a'}_a$. We can chose $g_{i'j'} = \text{diag}[g_{1'}, g_{2'}, g_{3'}]$, $h_{a'b'} = \text{diag}[h_{4'}, h_{5'}, \epsilon_{6'}]$ and $N_{i'}^{a'} = (N_{i'}^{3'} = w_{i'}, N_{i'}^{4'} = n_{i'}, N_{i'}^{5'} = 0)$ to be defined by any exact solution of type (10), or (14). The (pseudo) Finsler data f_{ij} , f_{ab} and ${}^c N_i^a = ({}^c N_i^3 = {}^c w_i, {}^c N_i^4 = {}^c n_i, 0)$ are with diagonal matrices, $f_{ij} = \text{diag}[f_1, f_2, f_3]$ and $f_{ab} = \text{diag}[f_4, f_5, \epsilon_6]$, if the generating function is of type $F = {}^1F(x^i, y^4) + {}^2F(x^i, y^5)$ for some homogeneous (respectively, on y^4 and y^5) functions 1F and 2F .¹⁵

The conditions (18) are satisfied for a diagonal representation for $e^{\alpha'}_{\alpha}$ if

$$\begin{aligned} e^{1'}_1 &= \pm \sqrt{\left| \frac{f_1}{g_{1'}} \right|}, & e^{2'}_2 &= \pm \sqrt{\left| \frac{f_2}{g_{2'}} \right|}, & e^{3'}_3 &= \pm \sqrt{\left| \frac{f_3}{g_{3'}} \right|}, \\ e^{4'}_4 &= \pm \sqrt{\left| \frac{f_4}{h_{4'}} \right|}, & e^{5'}_5 &= \pm \sqrt{\left| \frac{f_5}{h_{5'}} \right|}, & e^{6'}_6 &= \pm 1. \end{aligned}$$

For any fixed values f_i , f_a and ${}^c w_i, {}^c n_i$ and given $g_{i'}$ and $h_{a'}$, we can compute $w_{i'}$ and $n_{i'}$ as

$$\begin{aligned} w_{1'} &= \pm \sqrt{\left| \frac{g_{1'} f_4}{h_{4'} f_1} \right|} {}^c w_1, & w_{2'} &= \pm \sqrt{\left| \frac{g_{2'} f_4}{h_{4'} f_2} \right|} {}^c w_2, & w_{3'} &= \pm \sqrt{\left| \frac{g_{3'} f_4}{h_{4'} f_3} \right|} {}^c w_3, \\ n_{1'} &= \pm \sqrt{\left| \frac{g_{1'} f_5}{h_{5'} f_1} \right|} {}^c n_1, & n_{2'} &= \pm \sqrt{\left| \frac{g_{2'} f_5}{h_{5'} f_2} \right|} {}^c n_2, & n_{3'} &= \pm \sqrt{\left| \frac{g_{3'} f_5}{h_{5'} f_3} \right|} {}^c n_3, \end{aligned}$$

which defines solutions for (19).

So, any black ring metric and various types of (non) holonomic deformations can be expressed as a Sasaki type metric for a 3-d (pseudo) Finsler spacetime.¹⁶ If the coefficients of the nonholonomically deformed metric of type (10), or (14), are constrained to satisfy the

¹⁵Of course, we can work with arbitrary generating functions $F(x^i, y^a)$ but this will result in off-diagonal (pseudo) Finsler metrics in N-adapted bases, which would request a more cumbersome matrix calculus.

¹⁶In order to generate a homogeneous model on a total tangent bundle, we have to use another types of lifts of metrics from a base, see [31]. For simplicity, in this work, we consider gravitational models with Sasaki type lifts when the homogeneity is considered as a property only for the generating function but not obligatory for

conditions (11), we model stationary Finsler–Einstein spacetimes enabled with canonical d-connection structure. As a matter of principle, we can introduce Finsler like variables on any (pseudo) Riemannian spacetime, see details in [12–14]. In this case, we generate Einstein spacetimes with generic off-diagonal metrics, equivalently described in terms both of the Levi–Civita connection and the canonical d-connection with respect to the corresponding N-adapted frames, if the coefficients of such exact solutions are of type (17).

4 Vacuum (Pseudo) Finsler Ring Metrics

In this section, we construct exact solutions of (6), or (7), with $\lambda = 0$, defining vacuum (pseudo) Finsler models of ring solutions.

4.1 Stationary Vacuum Ansatz

4.1.1 Canonical d-Connection Vacuum Configurations

In general, because of generic nonholonomic and nonlinear character of Finsler type configurations, such metrics can not be obtained in a limit $\lambda \rightarrow 0$ for coefficients (11). It is necessary to apply the anholonomic deformation method of generating exact solutions from the very beginning to the vacuum Einstein equations for the canonical d-connection, $\hat{\mathbf{R}}_{\alpha\beta} = 0$, for an ansatz of type \mathbf{g} (10). The nontrivial coefficients of such an exact solution must satisfy the conditions

$$\begin{aligned} \epsilon_2 \varphi^{\bullet\bullet}(x^{\hat{k}}) + \epsilon_3 \varphi''(x^{\hat{k}}) &= 0; \\ h_4 &= \pm e^{-2^0 \varphi} \frac{(h_5^*)^2}{h_5} \quad \text{for a given } h_5(x^i, y^4), \quad \varphi = {}^0\varphi = \text{const}; \\ w_i &= w_i(x^k, y^4), \quad \text{for any such functions if } \lambda = 0; \\ n_i &= \begin{cases} {}^1n_i(x^k) + {}^2n_i(x^k) \int (h_5^*)^2 |h_5|^{-5/2} dy^4, & \text{if } n_i^* \neq 0; \\ {}^1n_i(x^k), & \text{if } n_i^* = 0. \end{cases} \end{aligned} \tag{20}$$

Metrics of this class define vacuum nonholonomic deformations of black rings if the coefficients of the primary ring metric $\check{\mathbf{g}}$ are introduced into a target vacuum metric as $g_i = \eta_i \check{g}_i$, $h_4 = \eta_4 \check{h}_4$ and $h_5 = \eta_5 \check{h}_5$ and ${}^1n_i(x^k) = C \check{R} \frac{1+y(x^3)}{F(x^3)}$.

It is not clear, in general, what physical importance may have solutions with data (20). In a similar manner as in the previous section, it is possible to extract nonholonomically polarized vacuum black ring solutions using decompositions on a small parameter ϵ like we have considered for the coefficients of ansatz (14). Such vacuum black rings can be also nonholonomically imbedded into solitonic backgrounds of type (16) and redefined in Finsler variables following transforms of type (18) and (19). On tangent bundles, such solutions are for the (pseudo) Finsler gravity with the canonical d-connection. This class of black rings contain some nonholonomic configurations which are topologically nontrivial and with broken local Lorentz symmetry. They can be also constructed in higher dimension extensions of (pseudo) Riemannian spaces but in those cases the anisotropic coordinates will not be of “velocity” type.

other geometric objects. It is possible to construct Finsler models when all fundamental geometric objects are rigorously subjected to the condition of homogeneity but they are very “nonlinear” in nature and subjected to more sophisticate conditions of nonholonomic constraints.

4.1.2 Levi–Civita Vacuum Configurations

It is necessary to impose additional constraints on the integral varieties in order to generate exact solutions of the Einstein equations for the Levi–Civita connection, i.e of ${}^1R_{\alpha\beta} = 0$. Such additional constraints can be of type

$$\begin{aligned}
 h_4 &= \pm 4 \left[\left(\sqrt{|h_5|} \right)^* \right]^2, \quad h_5^* \neq 0; \\
 w_2 w_3 \left(\ln \left| \frac{w_2}{w_3} \right| \right)^* &= w_3^\bullet - w_2', \quad w_i^* \neq 0; \\
 w_3^\bullet - w_2' &= 0, \quad \text{if } w_i^* = 0 \text{ and } w_1 = 0; \\
 {}^1n_2'(x^k) - {}^1n_3^\bullet(x^k) &= 0, \quad \text{if } n_1 = 0,
 \end{aligned}
 \tag{21}$$

for $e^{-2\phi} = 1$.

Decompositions on a small parameter ε result in ansatz of type (14) with the coefficients subjected additionally to the conditions (21). The rest of nontrivial coefficients can be chosen to be just those for a conformally transformed black ring metric (8) on a 6-d (pseudo) Riemannian space.

4.2 Physical Parameters for Nonholonomic Black Rings

A nonholonomic black ring configuration for a small ε , both for any zero and/or nonzero cosmological constant λ , can be characterized by physical parameters similar to those for holonomic black ring configurations given by formulas (2.6)–(2.10) in [25]. In the case of elliptic polarizations, we have to change $\check{R} \rightarrow \check{r}$ (15) and compute the physical parameters for the metric (14) considering that a corresponding observer is in a point in the v -part of spacetime and see a conformally transformed and locally anisotropically polarized black ring metric with deformed parameters on for direction x^2 (such an observer is in nonholonomic/“N-adapted” system of reference).

The corresponding effective locally anisotropic (ellipsoidal) mass, $M(\varepsilon, x^2)$, angular momentum, $J(\varepsilon, x^2)$, temperature, $T(\varepsilon, x^2)$, angular velocity, $\varpi(\varepsilon, x^2)$, and horizon area, $A_H(\varepsilon, x^2)$, are given by formulas

$$\begin{aligned}
 M &= \frac{3\pi}{4C} \frac{\check{\lambda}}{1-v} \check{r}^2(\varepsilon, x^2), & J &= \frac{\pi}{2C} \frac{\sqrt{\check{\lambda}(\check{\lambda}-v)(1+\check{\lambda})}}{(1-v)^2} \check{r}^3(\varepsilon, x^2), \\
 T &= \frac{1+v}{4\pi} \sqrt{\frac{1-\check{\lambda}}{v\check{\lambda}(1+\check{\lambda})}} \check{r}^{-1}(\varepsilon, x^2), & \varpi &= \sqrt{\frac{\check{\lambda}-v}{\check{\lambda}(1+\check{\lambda})}} \check{r}^{-1}(\varepsilon, x^2), \\
 A_H &= 8\pi \frac{v^{3/2} \sqrt{\check{\lambda}(1-\check{\lambda}^2)}}{(1-v)^2(1+v)} \check{r}^3(\varepsilon, x^2),
 \end{aligned}
 \tag{22}$$

where $\check{r}(\varepsilon, x^2) \rightarrow \check{R}$ for $\varepsilon \rightarrow 0$. Nevertheless, even in this limit such black ring configurations do not transform always into the well known homogeneous black ring solutions because the metric (14) may preserve its nonholonomic character with certain nontrivial N-connection coefficients (for instance, determined by certain nontrivial solitonic stationary configurations).

We can say that a nonholonomically polarized black ring with physical parameters (22) mimics a generic “off-diagonal” metric gravitational stationary configuration when an usual black ring is imbedded self-consistently into a (pseudo) Finsler/Riemannian background with gravitational polarizations like in an effective continuous media. For simplicity, we have chosen the simplest ellipsoidal polarization with small variable $\check{r}(\varepsilon, x^2)$ but it is possible to construct more general classes of solutions when the constants ν and/ or $\check{\lambda}$ are also polarized. Such nonholonomic configurations with ring topology can be also characterized by locally anisotropic physical parameters (see some similar examples related to black torus and torus-ellipsoid configurations in [7]).

5 Concluding Remarks

To summarize we have shown how black ring solutions can be constructed in (pseudo) Finsler and nonholonomic (pseudo) Riemannian spacetimes. We used the anholonomic deformation method of constructing exact solutions with generic off-diagonal metrics in Einstein gravity and generalizations [6, 7, 12, 13]. The Finsler–Einstein spacetimes with anisotropically polarized constants and/or nonholonomic imbedding into nontrivial backgrounds are interesting configurations to study from both Finsler gravity and extra dimension gravity perspectives. Such objects can be described by asymptotically flat gravitational configurations with a non spherical horizon topology generated by nonholonomic deformations of standard black ring metrics.

The first class of locally anisotropic stationary solutions was considered for the Einstein equations with nontrivial constant in Finsler gravity and six dimensional extension of general relativity. They were generated by nonholonomic deformations of the black ring metric imbedded into corresponding (pseudo) Finsler/Riemannian manifolds. For the new classes of Finsler–Einstein spacetimes, the dependence on the cosmological constant is generic nonlinear and non-integrable. It is not clear what type of physical interpretation such solutions may have in general even it is obvious that they contain as a corresponding subclasses various types of black ring solutions in braneworld gravity, string gravity with nontrivial cosmological constant etc. Nevertheless, it is possible to extract physically important solutions with nontrivial topology of horizon and gravitational polarizations of cosmological constants using a procedure of extracting solutions depending on a small parameter, for instance, characterizing small “ellipsoidal” deformations.

The second class of Finsler type solutions are for vacuum configurations defined by generic off-diagonal metrics and corresponding nonlinear and linear connection structures. They also consist examples of Finsler–Einstein spacetimes but, in general, they can not be generated in the trivial limit of zero cosmological constant.

It might be possible to gain more insight on viability of gravity Finsler type and extra dimension theories using properties of black hole/ring and nonlinear wave solutions in such models and comparing them to similar ones in Einstein gravity and string/brane gravity. Picturing such objects as nonholonomic deformations of already studied physical gravitational systems but with additional polarizations of physical constants, deformed symmetries and imbedding, for instance, into solitonic backgrounds, we provide realistic physical interpretations for a very general class of generic off-diagonal metrics in general relativity and extra dimension gravity.

The question of stability of the Finsler like black ring configurations can be solved explicitly for certain limits of small nonholonomic deformations by using the analysis performed

for usual holonomic black rings [25] and black ellipsoid solutions [26]. For general non-holonomic transforms of a stable black ring metric, the target solution may be both stable or unstable with an undetermined physical status.

As we discussed in the Introduction, the results of this work bridges the gap between rather different directions of research: new methods of constructing exact solutions in gravity theories, Finsler gravity and extra dimension generalizations of general relativity. Our approach allows to study solutions on tangent bundle and/or for nonholonomic manifold spacetime models. The constructions on tangent bundles are related to theories with violation of local Lorentz symmetry and so-called “non-standard” models of gravity, see a conventional classification in [13]. An approach to (pseudo) Finsler gravity and generalizations being compatible with the modern paradigm of standard physics can be elaborated using the geometric formalism of nonholonomic manifolds and associated nonlinear connection structures.

It would be interesting to see whether the classes of solutions studied in this work, and in the partner paper [14], can be generalized for noncommutative gravity [6] and Ricci flow physical models [28]. Another very important extension of our results would be to encompass gauge gravity and warped locally anisotropic geometries. Last, but not least, it would be interesting to use the anholonomic deformation method to construct cosmological solutions parametrized by generic off-diagonal metrics and possessing generalized symmetries and anisotropic polarizations of fundamental physical constants.

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